

# Proof of the Bonheure-Noris-Weth conjecture on oscillatory radial solutions of Neumann problems

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**Abstract.** Let  $B_1$  be the unit ball in  $\mathbb{R}^N$  with  $N \geq 2$ . Let  $f \in C^1([0, \infty), \mathbb{R})$ ,  $f(0) = 0$ ,  $f(\beta) = \beta$ ,  $f(s) < s$  for  $s \in (0, \beta)$ ,  $f(s) > s$  for  $s \in (\beta, \infty)$  and  $f'(\beta) > \lambda_k^r$ . D. Bonheure, B. Noris and T. Weth [Ann. Inst. H. Poincaré Anal. Non Linéaire 29(4) (2012)] proved the existence of nondecreasing, radial positive solutions of the semilinear Neumann problem

$$-\Delta u + u = f(u) \text{ in } B_1, \quad \partial_\nu u = 0 \text{ on } \partial B_1$$

for  $k = 2$ , and they conjectured that there exists a radial solution with  $k$  intersections with  $\beta$  provided that  $f'(\beta) > \lambda_k^r$  for  $k > 2$ . In this paper, we show that the answer is yes.

**Keywords.** Bonheure-Noris-Weth conjecture; Neumann problem; oscillatory radial solutions; bifurcation.

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## 1 Introduction

Let  $B_1$  be the unit ball in  $\mathbb{R}^N$  with  $N \geq 2$ . Very recently, D. Bonheure, B. Noris and T. Weth [1] proved the existence of nondecreasing, radial positive solutions of the semilinear Neumann problem

$$\begin{cases} -\Delta u + u = f(u) & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ \partial_\nu u = 0 & \text{on } \partial B_1 \end{cases} \quad (1.1)$$

under the assumptions:

- (f1)  $f \in C^1([0, \infty), \mathbb{R})$ ,  $f(0) = 0$  and  $f$  is nondecreasing;
- (f2)  $f'(0) = \lim_{s \rightarrow 0^+} \frac{f(s)}{s} = 0$ ;
- (f3)  $\liminf_{s \rightarrow +\infty} \frac{f(s)}{s} > 1$ ;
- (f4) there exists  $\beta > 0$  such that  $f(\beta) = \beta$  and

$$f'(\beta) > \lambda_2^r. \quad (1.2)$$

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Here  $\lambda_k^r$  is the  $k$ -th radial eigenvalue of  $-\Delta + I$  in the unit ball with Neumann boundary conditions.

It is easy to see that  $u \equiv \beta$  is a constant solution of (1.1), and there exists nonlinearity  $f$  satisfying (f1) – (f3) such that the problem (1.1) only admits this constant solution, see [1, Proposition 4.1]. For the existence of nonconstant radial solutions, they obtained the following result by variational argument.

**Theorem A.** Assume (f1) – (f4). Then there exists at least one nonconstant increasing radial solution of (1.1).

They raised the question whether it is possible to construct radial solutions with a given number of intersections with  $\beta$  provided that  $f'(\beta)$  is sufficiently large. More precisely, they conjectured that there exists a radial solution with  $k$  intersections with  $\beta$  provided that  $f'(\beta) > \lambda_k^r$ .

The purpose of the present paper is to show that the answer to the above question is yes! The proof is based upon the unilateral global bifurcation theorem [4, 5, 8]. The condition  $f'(0) = 0$  and the monotonic condition in (f1) seem unduly restrictive. We shall make the following assumptions:

(A1)  $f \in C^1([0, \infty), \mathbb{R})$ ,  $f(0) = 0$ ;

(A2)  $f_{+\infty} := \lim_{s \rightarrow +\infty} \frac{f(s)}{s} < \infty$ ;

(A3) there exists  $\beta > 0$  such that

$$f(\beta) = \beta, \quad f(s) < s \text{ for } s \in (0, \beta), \quad f(s) > s \text{ for } s \in (\beta, \infty)$$

and

$$f'(\beta) > \lambda_k^r, \quad \text{for some } k \geq 2;$$

(A4)  $[f(s + \beta) - (s + \beta)]s > 0$ ,  $s \in (-\beta, 0) \cup (0, \infty)$ .

The main result of this paper is the following

**Theorem 1.1** Assume (A1)-(A3). Then for each  $j \in \{2, \dots, k\}$ , (1.1) has two nonconstant radial solutions  $u_j^+$  and  $u_j^-$  such that  $u_j^+ - \beta$  changes sign exactly  $k - j + 1$  times in  $(0, 1)$  and is positive near 0, and  $u_j^- - \beta$  changes sign exactly  $k - j + 1$  times in  $(0, 1)$  and is negative near 0. Moreover, if (A4) holds, then  $u_2^+$  is decreasing in  $[0, 1]$  and  $u_2^-$  is increasing in  $[0, 1]$ .

For other results on the existence of radial solutions of nonlinear Neumann problems, see [2, 10, 14].

The rest of the paper is organized as follows. In Section 2 we study the spectrum structure of the linear Neumann problem

$$\begin{cases} -\Delta u(x) = \mu a(|x|)u(x) & \text{in } B_1, \\ \partial_\nu u = 0 & \text{on } \partial B_1, \end{cases}$$

where  $a \in C[0, 1]$  satisfies  $a(r) > 0$  for  $r \in [0, 1]$ . In Section 3, we introduce some functional setting and state some preliminary bifurcation results on abstract operator equations. Finally in Section 4 we prove our main results on the existence of nonconstant radial solutions by applying the well-known unilateral bifurcation theorem due to Dancer [4, 5].

## 2 Eigenvalues of linear eigenvalue problems

Let us consider the linear eigenvalue problem

$$\begin{cases} -\Delta u(x) = \mu a(|x|)u(x) & \text{in } B_1, \\ \partial_\nu u = 0 & \text{on } \partial B_1, \end{cases} \quad (2.1)$$

where  $a \in C[0, 1]$  satisfies

$$a(r) > 0, \quad r \in [0, 1]. \quad (2.2)$$

**Theorem 2.1** Assume that (2.2) is fulfilled. Then the radial eigenvalues of (2.1) are as follows:

$$0 = \mu_0^r < \mu_1^r < \mu_2^r < \cdots \rightarrow \infty. \quad (2.3)$$

Moreover, for each  $k \in \mathbb{N}^* := \{0, 1, 2, \dots\}$ , the radial eigenvalue  $\mu_k^r$  is simple, and the radial eigenfunction  $\psi_k$ , being regarded as a function of  $r$ , possesses exactly  $k$  simple zeros in  $[0, 1]$ , and  $\psi_k$  is radially monotone if and only if  $k \in \{0, 1\}$ .

It is easy to see that Theorem 2.1 is an immediate consequence of the following results on singular Sturm-Liouville problems.

**Theorem 2.2** Assume that (2.2) is fulfilled. Then the eigenvalues of the problem

$$\begin{cases} -u''(r) - \frac{N-1}{r}u'(r) = \mu a(r)u(r), & r \in (0, 1), \\ u'(0) = 0 = u'(1) \end{cases} \quad (2.4)$$

are as follows:

$$0 = \mu_0^r < \mu_1^r < \mu_2^r < \cdots \rightarrow \infty.$$

Moreover, for each  $k \in \mathbb{N}^*$ ,  $\mu_k^r$  is simple, and the eigenfunction  $\psi_k$  possesses exactly  $k$  simple zeros in  $[0, 1]$ , and  $\psi_k$  is monotone if and only if  $k \in \{0, 1\}$ .

To prove Theorem 2.2, we need several basic lemmas.

**Lemma 2.1** Assume that  $\tilde{f} \in C([0, \infty) \times [0, \infty))$  is Lipschitz continuous in  $u$  on  $[0, \infty)$ . Then for given  $\zeta \in (0, \infty)$ , the initial value problem

$$\begin{cases} -u''(r) - \frac{N-1}{r}u'(r) = \tilde{f}(r, u(r)), & r \in (0, \infty), \\ u'(0) = 0, \\ u(0) = \zeta \end{cases} \quad (2.5)$$

has a unique solution  $u$  defined on  $[0, \infty)$ . Moreover, all of zeros of  $u$  are simple.

**Proof.** According to [15, Existence and uniqueness Theorem XIII in §6 of Chapter II], for given  $b > 0$ , the initial value problem

$$\begin{cases} -u''(r) - \frac{N-1}{r}u'(r) = \tilde{f}(r, u(r)), & r \in (0, b], \\ u'(0) = 0, \\ u(0) = \zeta \end{cases}$$

has exactly one solution  $u \in C^2[0, b]$ . Notice the equation in (2.5) is non-singular for  $r \geq b$ , it is evident that  $u$  can be extended to  $[0, \infty)$ . Since  $\tilde{f} \in C([0, \infty) \times [0, \infty))$  is Lipschitz continuous in  $u$  on  $[0, \infty)$ , the uniqueness part can be deduced by the same method in the Appendix in [11].

All of zeros of  $u$  are simple since for any zero point  $\tau$  of  $u$ , the initial value problem

$$\begin{cases} -u''(r) - \frac{N-1}{r}u'(r) = f(r, u(r)), & r \in (0, \infty), \\ u(\tau) = 0 = u'(\tau) \end{cases}$$

has only trivial solution  $u \equiv 0$ . □

**Lemma 2.2** Assume that  $a \in C([0, \infty), (0, \infty))$  and there exist two positive constants  $a_1$  and  $a_2$ , such that

$$a_1 \leq a(r) \leq a_2, \quad r \in [0, \infty).$$

Let  $u$  be a solution of the problem

$$\begin{cases} -u''(r) - \frac{N-1}{r}u'(r) = \mu a(r)u(r), & r \in (0, \infty), \\ u'(0) = 0, \\ u(0) = \zeta \end{cases} \quad (2.6)$$

with  $\mu > 0$  and  $\zeta > 0$ . Then  $u$  has a sequence of zeros  $\{\tau_n\} \subset (0, \infty)$  with

$$\tau_n \rightarrow \infty \quad \text{as } n \rightarrow \infty. \quad (2.7)$$

**Proof.** From [15, XVIII in §27 of Chapter VI], the solution  $y$  of the initial value problem

$$(r^{N-1}y')' + r^{N-1}y = 0, \quad y(0) = 1, \quad y'(0) = 0 \quad (2.8)$$

oscillates. Denote the zeros of  $y$  by  $\xi_0 < \xi_1 < \xi_2 < \dots$ . Then

$$\xi_{n+1} - \xi_n \rightarrow \pi, \quad \text{as } n \rightarrow \infty. \quad (2.9)$$

Let  $\gamma_j = \sqrt{\mu a_j}$  for  $j = 1, 2$ . Let

$$u_j(r) = y(\gamma_j r), \quad j = 1, 2.$$

Then  $u_j$  is the unique solution of the initial value problem

$$(r^{N-1}u')' + \gamma_j^2 r^{N-1}u = 0, \quad u(0) = 1, \quad u'(0) = 0, \quad (2.10)$$

and for  $j = 1, 2$ ,  $u_j$  oscillates and it has a sequence of zeros  $\frac{\xi_0}{\gamma_j} < \frac{\xi_1}{\gamma_j} < \frac{\xi_2}{\gamma_j} < \dots$ . Combining this with the Sturm-Picone Theorem ( see [15, §27 of Chapter VI]), it deduces that the solution  $u$  of (2.6) oscillates.  $\square$

**Lemma 2.3** Assume that  $a \in C([0, \infty), (0, \infty))$ . Let  $u$  be a solution of the problem (2.6) with  $\mu > 0$  and  $\zeta > 0$ . Let  $r_1, r_2$  be any two consecutive zeros of  $u'$  in  $[0, \infty)$  with  $r_1 < r_2$ . Then  $u$  has one and only one zero in  $(r_1, r_2)$ .

**Proof.**  $-u''(r) - \frac{N-1}{r}u'(r) = \mu a(r)u(r)$  can be rewritten as

$$-(r^{N-1}u'(r))' = \mu a(r)r^{N-1}u(r). \quad (2.11)$$

Integrating from  $r_1$  to  $r$ , we get

$$u'(r) = -\mu \int_{r_1}^r \left(\frac{t}{r}\right)^{N-1} a(t)u(t)dt,$$

and accordingly,

$$0 = u'(r_2) = -\mu \int_{r_1}^{r_2} \left(\frac{t}{r_2}\right)^{N-1} a(t)u(t)dt,$$

which implies that  $u$  has at least one zero in  $(r_1, r_2)$ .

Suppose on the contrary that  $u$  has two zeros  $z_1, z_2 \in (r_1, r_2)$ . Then there exists  $z^* \in (z_1, z_2) \subset (r_1, r_2)$ , such that  $u'(z^*) = 0$ . However, this is a contradiction.  $\square$

**Lemma 2.4** Assume that  $a \in C([0, \infty), (0, \infty))$ . Let  $u$  be a solution of (2.6) with  $\mu > 0$  and  $\zeta > 0$ . Let  $\tau_1, \tau_2$  be any two consecutive zeros of  $u$  in  $(0, \infty)$  with  $\tau_1 < \tau_2$ . Then  $u'$  has one and only one zero in  $(\tau_1, \tau_2)$ .

**Proof.** Obviously,  $u'$  has at least one zero  $r_1$  in  $(\tau_1, \tau_2)$ .

Without loss of generality, we may assume that  $u(r) > 0$  in  $(\tau_1, \tau_2)$ . It follows from

$$-u''(r) - \frac{N-1}{r}u'(r) = \mu a(r)u(r) \quad (2.12)$$

that

$$u''(r_1) < 0,$$

which implies that  $u$  is concave up near  $r = r_1$ .

Suppose on the contrary that there exists  $r_* \in (\tau_1, \tau_2)$  with  $r_* \neq r_1$  such that  $u'(r_*) = 0$ . Then by the same argument, we get

$$u''(r_*) < 0$$

This together with  $u''(r_1) < 0$  imply that there exists  $\hat{r} \in (\min\{r_1, r_*\}, \max\{r_1, r_*\})$  such that  $u$  attains a local minimum at  $\hat{r}$ , and

$$u(\hat{r}) > 0, \quad u'(\hat{r}) = 0, \quad u''(\hat{r}) \geq 0,$$

which contradicts (2.12). Therefore,  $u'$  has only one zero in  $(\tau_1, \tau_2)$ .  $\square$

**Lemma 2.5** Assume that  $a \in C([0, \infty), (0, \infty))$  with  $a(r) \geq a_0 > 0$  in  $(0, \infty)$ . Let  $u$  be a solution of (2.6) with  $\zeta > 0$  and  $\mu > 0$ . Let  $\tau_k(\mu)$  and  $r_k(\mu)$  be the  $k$ -th positive zero of  $u$  and  $u'$ , respectively. Then

- (1) For given  $k \in \mathbb{N}$ ,  $\tau_k(\mu)$  is strictly decreasing in  $(0, \infty)$ ;
- (2) For given  $k \in \mathbb{N}$ ,  $r_k(\mu)$  is strictly decreasing in  $(0, \infty)$ .

**Proof.** (1) For fixed  $k > 1$ ,  $\tau_k(\mu)$  is strictly decreasing in  $\mu$ , which is an immediate consequence of the well-known Sturm Separation Theorem [15, P. 272] since the differential equation

$$-u'' - \frac{N-1}{r}u' = \mu a(r)u$$

is non-singular for  $r \geq \tau_1(\mu)$ . So, we only need to show that  $\tau_1(\mu)$  is strictly decreasing in  $(0, \infty)$ .

Let  $\tau_1(\mu)$  be the first zero of the solution  $u$  of the initial value problem

$$\begin{cases} -(r^{N-1}u'(r))' = \mu r^{N-1}a(r)u(r), & r \in (0, \infty), \\ u'(0) = 0, \\ u(0) = \zeta. \end{cases} \quad (2.13)$$

Let  $\tau_1(\mu^*)$  be the first zero of the solution  $v$  of the initial value problem

$$\begin{cases} -(r^{N-1}v'(r))' = \mu^* r^{N-1}a(r)v(r), & r \in (0, \infty), \\ v'(0) = 0, \\ v(0) = \zeta. \end{cases} \quad (2.14)$$

We only need to show that

$$\tau_1(\mu) > \tau_1(\mu^*) \quad \text{if } \mu^* > \mu. \quad (2.15)$$

Suppose on the contrary that  $\tau_1(\mu) \leq \tau_1(\mu^*)$ . Then

$$v(r) > 0, \quad r \in [0, \tau_1(\mu)); \quad v'(\tau_1(\mu)) < 0. \quad (2.16)$$

Multiplying the equations in (2.13) and (2.14) by  $v$  and  $u$ , respectively, and integrating from 0 to  $\tau_1(\mu)$ , we get

$$-(\tau_1(\mu))^{N-1}v(\tau_1(\mu))u'(\tau_1(\mu)) = (\mu - \mu^*) \int_0^{\tau_1(\mu)} r^{N-1}a(r)u(r)v(r)dr.$$

However, this is impossible from (2.16) and the fact

$$u(r) > 0, \quad r \in [0, \tau_1(\mu)); \quad u'(\tau_1(\mu)) < 0.$$

Therefore, (2.15) is valid.  $\square$

(2) Using the similar method to treat (2.15) and the fact  $\tau_k(\mu)$  is strictly decreasing in  $(0, \infty)$ , it is not difficult to show that  $r_k(\mu)$  is strictly decreasing for  $\mu \in (0, \infty)$ .

Let  $u$  be the solution of (2.13) and  $r_k(\mu)$  be the  $k$ -th positive zero of  $u'$ . Then

$$\tau_k(\mu) < r_k(\mu) < \tau_{k+1}(\mu).$$

Without loss of generality, we may assume that

$$u'(r) < 0, \quad r \in (r_{k-1}(\mu), r_k(\mu)); \quad u(r) < 0, \quad r \in (\tau_k(\mu), r_k(\mu)). \quad (2.17)$$

(The other cases can be proved by the similar method.) Let  $v$  be the solution of (2.14) and  $r_k(\mu^*)$  be the  $k$ -th positive zero of  $v'$ . Then it follows from (2.17) that

$$v'(r) < 0, \quad r \in (r_{k-1}(\mu^*), r_k(\mu^*)). \quad (2.18)$$

Suppose on the contrary that there exist some  $k$  and some  $\mu, \mu^*$  with  $\mu < \mu^*$ , such that

$$r_k(\mu) \leq r_k(\mu^*). \quad (2.19)$$

Combining this with the fact that  $\tau_k(\mu)$  is strictly decreasing in  $\mu$  and using (2.17), it follows that

$$v'(r) < 0, \quad r \in [\tau_k(\mu), r_k(\mu)]; \quad v(r) < 0, \quad r \in [\tau_k(\mu), r_k(\mu)].$$

Multiplying the equation in (2.14) by  $u$  and the equation in (2.13) by  $v$  and integrating from  $\tau_k(\mu)$  to  $r_k(\mu)$ , we get

$$\begin{aligned} & (\tau_k(\mu))^{N-1}v(\tau_k(\mu))u'(\tau_k(\mu)) + (r_k(\mu))^{N-1}v'(r_k(\mu))u(r_k(\mu)) \\ &= (\mu - \mu^*) \int_{\tau_k(\mu)}^{r_k(\mu)} r^{N-1}a(r)u(r)v(r)dr. \end{aligned}$$

This together with the signs of  $u, u', v, v'$  at  $\tau_k(\mu)$  and  $r_k(\mu)$  imply that (2.19) is impossible.  $\square$

**Proof of Theorem 2.2** Let  $u(r; \zeta, \mu)$  be the unique solution of (2.6). For  $k \in \mathbb{N}$ . Let  $\mu_k^r$  be such that  $u'(1; \zeta, \mu_k^r) = 0$  and  $u(r; \zeta, \mu_k^r)$  has exactly  $k$  zeros in  $(0, 1)$ .

Let

$$\psi_k(r) := u(r; \zeta, \mu_k^r), \quad r \in [0, 1].$$

Then Lemmas 2.1-2.5 guarantee the desired results. In particular,

$$\psi_0(r) \equiv \zeta; \quad \psi_1(r) \text{ is monotone on } r \in (0, 1).$$

□

**Lemma 2.6** Let  $\{(\mu_n, y_n)\}$  be a sequence of solutions of the problem

$$-(r^{N-1}y_n')' = \mu_n r^{N-1}g(y_n), \quad y_n'(0) = y_n'(1) = 0, \quad (2.20)$$

where  $|\mu_n| \leq \hat{\mu}$  ( $\hat{\mu}$  is a positive constant),  $g : \mathbb{R} \rightarrow \mathbb{R}$  satisfies

$$|g(s)| \leq L_0|s| \quad \text{for some constant } L_0 > 0.$$

Then  $\|y_n'\|_\infty \rightarrow \infty$  as  $n \rightarrow \infty$  implies  $\|y_n\|_\infty \rightarrow \infty$  as  $n \rightarrow \infty$ .

**Proof.** Assume on the contrary that  $\|y_n\|_\infty \not\rightarrow \infty$  as  $n \rightarrow \infty$ . Then, after taking a subsequence and relabeling, if necessary, it follows that

$$\|y_n\|_\infty \leq M_0 \quad (2.21)$$

for some  $M_0 > 0$ . From (2.20), we get

$$y_n'(r) = -\mu_n \int_0^r \left(\frac{s}{r}\right)^{N-1} g(y_n(s)) ds,$$

which implies that

$$\|y_n'\|_\infty \leq \hat{\mu} L_0 \cdot \|y_n\|_\infty \leq \hat{\mu} M_0 L_0.$$

However, this is a contradiction. □

### 3 Functional setting and preliminary properties

The main point to prove Theorem 1.1 consists in using the unilateral global bifurcation theorem of [4, 5, 8]

Let  $E$  be a real Banach space with norm  $\|\cdot\|$ .  $\mathcal{E}$  will denote  $E \times \mathbb{R}$ . Let the mapping  $\mathcal{G} : \mathcal{E} \rightarrow E$  satisfy

*Assumption  $\mathfrak{A}$ :* if  $\mathcal{G}(0, \lambda) = 0$  for  $\lambda \in \mathbb{R}$ ,  $\mathcal{G}$  is completely continuous and

$$\mathcal{G}(x, \lambda) = \lambda Lx + H(x, \lambda),$$



where  $L$  is a completely continuous linear operator on  $E$  and  $\|H(x, \lambda)\|/\|x\| \rightarrow 0$  uniformly on bounded subsets of  $\mathbb{R}$  as  $\|x\| \rightarrow 0$ .

Define  $\Phi(\lambda) : E \rightarrow E$  by  $\Phi(\lambda)(x) = x - \mathcal{G}(x, \lambda)$  and define  $\mathfrak{L}$  to be the closure of  $\{(x, \lambda) \in \mathcal{E} : x = \mathcal{G}(x, \lambda), x \neq 0\}$  in  $\mathcal{E}$ . Then (cp. Rabinowitz [13])  $\mathfrak{L} \cap (\{0\} \times \mathbb{R}) \subseteq \{0\} \times r(L)$ , where  $r(L)$  denotes the real characteristic value of  $L$ . If  $\mu \in r(L)$ , define  $C_\mu$  to be the component of  $\mathfrak{L}$  containing  $(0, \mu)$ .

Assume now that  $\mu \in r(L)$  such that  $\mu$  has multiplicity 1. Suppose that  $v \in E \setminus \{0\}$  and  $l \in E^*$  such that

$$v = \mu Lv, \quad l = \mu L^* l,$$

(where  $L^*$  is the adjoint of  $L$ ) and  $l(v) = 1$ . If  $y \in (0, 1)$ , define

$$K_y = \{(u, \lambda) \in \mathcal{E} : |l(u)| > y\|u\|\},$$

$$K_y^+ = \{(u, \lambda) \in \mathcal{E} : l(u) > y\|u\|\}, \quad K_y^- = \{(u, \lambda) \in \mathcal{E} : l(u) < -y\|u\|\}.$$

By [13, Lemma 1.24], there exists an  $S > 0$  such that

$$(\mathcal{L} \setminus \{(\mu, 0)\}) \cap \bar{\mathcal{E}}_S(\mu) \subseteq K_y,$$

where  $\mathcal{E}_S(\mu) = \{(u, \lambda) \in \mathcal{E} : \|u\| + |\lambda - \mu| < S\}$  and  $\bar{\mathcal{E}}_S(\mu)$  denotes closure of  $\mathcal{E}_S(\mu)$ . For  $0 < \epsilon \leq S$  and  $\nu = \pm$ , define  $D_{\mu, \epsilon}^\nu$  to be the component of  $\{(0, \mu)\} \cup (\mathfrak{L} \cap \bar{\mathcal{E}}_\epsilon(\mu) \cap K_y^\nu)$  containing  $(0, \mu)$ ,  $C_{\mu, \epsilon}^\nu$  to be the component of  $\overline{C_\mu \setminus \mathcal{D}_{\mu, \epsilon}^{-\nu}}$  containing  $(0, \mu)$  (where  $-\nu$  is interpreted in the natural way), and  $C_{\mu, \nu}$  to be the closure of  $\bigcup_{S \geq \epsilon > 0} C_{\mu, \epsilon}^\nu$ . Then  $C_{\mu, \nu}$  is connected and, by [5],  $C_\mu = C_{\mu, +} \cup C_{\mu, -}$ . By [13, Lemma 1.24], the definition of  $C_{\mu, \nu}$  is independent of  $y$ .

**Theorem 3.1** [5, Theorem 2] Either  $C_{\mu, +}$  and  $C_{\mu, -}$  are both unbounded or

$$C_{\mu, +} \cap C_{\mu, -} \neq \{(0, \mu)\}.$$

## 4 Proof of the Main Results

Let  $X := \{u \in C^1[0, 1] : u'(0) = u'(1) = 0\}$ . Then it is a Banach space under the norm

$$\|u\|_X = \max\{\|u\|_\infty, \|u'\|_\infty\}.$$

We shall prove that the first choice of the alternative of Theorem 3.1 is the only possibility.

In what follows, we use the terminology of Rabinowitz [13]. Let  $S_{k, +}$  denote the set of functions in  $X$  which have exactly  $k - 1$  interior nodal (i.e. non-degenerate) zeros in  $(0, 1)$  and

are positive near  $r = 0$ , set  $S_{k,-} = -S_{k,+}$ , and  $S_k = S_{k,+} \cup S_{k,-}$ . Finally, let  $\Phi_{k,\pm} = \mathbb{R} \times S_{k,\pm}$  and  $\Phi_k = \mathbb{R} \times S_k$  under the product topology.

Let us consider the problem

$$\begin{cases} -\Delta u + u = f(u) & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ \partial_\nu u = 0 & \text{on } \partial B_1, \end{cases} \quad (4.1)$$

which is equivalent to

$$\begin{cases} -u'' - \frac{N-1}{r}u' + u = f(u), & r \in (0, 1), \\ u > 0, & r \in [0, 1], \\ u'(0) = u'(1) = 0. \end{cases} \quad (4.2)$$

Let

$$v := u - \beta.$$

Then (4.2) can be rewritten as

$$\begin{cases} -v'' - \frac{N-1}{r}v' + v = f(v + \beta) - \beta, & r \in (0, 1), \\ v > -\beta, & r \in [0, 1], \\ v'(0) = v'(1) = 0. \end{cases} \quad (4.3)$$

Let

$$h(s) := \begin{cases} f(s + \beta) - \beta, & s \geq -\beta, \\ -\beta, & s < -\beta. \end{cases}$$

Then

$$h(v) = h'(0)v + \xi(v) = f'(\beta)v + \xi(v), \quad h'(0) = f'(\beta),$$

and

$$\xi'(0) := \lim_{v \rightarrow 0} \frac{\xi(v)}{v} = 0. \quad (4.4)$$

Thus, to study the  $S_{k,\nu}$ -solutions of (4.3), let us consider the auxiliary problem

$$\begin{cases} -v'' - \frac{N-1}{r}v' + v = \lambda f'(\beta)v + \lambda \xi(v), & r \in (0, 1), \\ v > -\beta, & r \in [0, 1], \\ v'(0) = v'(1) = 0. \end{cases} \quad (4.5)$$

For  $e \in X$ , let  $Te$  be the unique solution of the problem

$$\begin{cases} -z'' - \frac{N-1}{r}z' + z = e, & r \in (0, 1), \\ z'(0) = z'(1) = 0. \end{cases}$$

Then the map  $T : X \rightarrow X$  is completely continuous, and

$$r(T) = \{\lambda_j^r \mid \lambda_j^r = \mu_{j-1}^r + 1, j = 1, 2, \dots\}.$$

Here  $r(T)$  denotes the real characteristic value of  $T$ . Obviously (4.5) is equivalent to

$$v = \lambda T(f'(\beta)v) + \lambda T\xi(v), \quad (4.7)$$

$$v > -\beta. \quad (4.8)$$

To show that (4.5) has a  $S_{k,\nu}$ -solution, let us consider the auxiliary problem (4.7)-(4.8) as a bifurcation problem from the trivial solution  $v \equiv 0$ . Furthermore, we have from (4.4) that

$$\frac{\|T\xi(v)\|_X}{\|v\|_X} \leq \|T\|_{X \rightarrow X} \max \left\{ \frac{\|\xi(v)\|_\infty}{\|v\|_X}, \frac{\|\xi'(v)v'\|_\infty}{\|v\|_X} \right\} \rightarrow 0 \quad \text{as } \|v\|_X \rightarrow 0.$$

Now the Dancer's unilateral global bifurcation theorem for (4.7) can be stated as follows:

Let

$$\mathfrak{L} := \overline{\{(\lambda, v) \in (0, \infty) \times X : (\lambda, v) \text{ satisfies (4.7), } v \neq 0\}}^X.$$

For  $\lambda_k^r \in r(T)$ , define  $C_k$  to be the component of  $\mathfrak{L}$  containing  $(\frac{\lambda_k^r}{f'(\beta)}, 0)$ . Then

$$C_k := C_{k,+} \cup C_{k,-},$$

where

$$C_{k,\nu} := C_{\lambda_k^r/f'(\beta),\nu} \quad \nu \in \{+, -\},$$

see Section 3 for detail. Now the Dancer's unilateral global bifurcation theorem yields that either  $C_{k,+}$  and  $C_{k,-}$  are both unbounded or

$$C_{k,+} \cap C_{k,-} \neq \{(\frac{\lambda_k^r}{f'(\beta)}, 0)\}. \quad (4.9)$$

From (A1), it follows that if  $(\lambda, v)$  is a solution of

$$\begin{cases} -v'' - \frac{N-1}{r}v' + v = \lambda h(v), \\ v(\tau) = v'(\tau) = 0 \end{cases}$$

for some  $\tau \in (0, \infty)$ , then  $v \equiv 0$ . This implies that

$$C_{k,+} \subset \left( \Phi_{k,+} \cup \{(\frac{\lambda_k^r}{f'(\beta)}, 0)\} \right).$$

Clearly, if (4.9) holds, then there exists  $(\lambda_*, v_*) \in C_{k,+} \cap C_{k,-}$ , such that  $(\lambda_*, v_*) \neq (\frac{\lambda_k^r}{f'(\beta)}, 0)$ , and  $v_* \in S_{k,+} \cap S_{k,-}$ , which contradicts the definition of  $S_{k,+}$  and  $S_{k,-}$ .

Furthermore, we get

**Lemma 4.1.** For given  $k \geq 2$ ,  $C_{k,+}$  and  $C_{k,-}$  are both unbounded, and  $(\frac{\lambda_k^r}{f'(\beta)}, 0)$  bifurcates two unbounded components  $C_{k,+}$  and  $C_{k,-}$  of solutions to problem (4.7), such that

$$(C_{k,+} \setminus \{(\frac{\lambda_k^r}{f'(\beta)}, 0)\}) \subseteq \Phi_{k,+}, \quad (C_{k,-} \setminus \{(\frac{\lambda_k^r}{f'(\beta)}, 0)\}) \subseteq \Phi_{k,-}.$$

**Lemma 4.2.** Let  $(\lambda, v) \in C_{k,\nu}$  with  $\lambda \in [0, 1]$ . Then

$$v(r) > -\beta, \quad r \in [0, 1]. \quad (4.10)$$

**Proof.** Suppose on the contrary that there exists  $x_0 \in [0, 1]$  such that

$$v(x_0) = \min_{r \in [0, 1]} v(r) = -\beta.$$

Then there exists  $r_0 \in [0, 1]$  such that either

$$v(r_0) = 0, \quad v(r) < 0 \text{ for } r \in [x_0, r_0), \quad v'(r) > 0 \text{ for } r \in (x_0, r_0]; \quad (4.11)$$

or

$$v(r_0) = 0, \quad v(r) < 0 \text{ for } r \in (r_0, x_0], \quad v'(r) < 0 \text{ for } r \in [r_0, x_0). \quad (4.12)$$

We only deal with the case (4.11), the case (4.12) can be treated by the similar way.

By (A1)-(A3), there exists  $m \geq 0$  such that  $h(s) + ms$  is monotone increasing in  $s$  for  $s \in [-\beta, +\infty)$ . Then

$$-v'' - \frac{N-1}{r}v' + v + \lambda mv = \lambda[h(v) + mv], \quad r \in (0, 1],$$

and, since

$$-(-\beta)'' - \frac{N-1}{r}(-\beta)' + (-\beta) + \lambda m(-\beta) \leq \lambda[h(-\beta) + m(-\beta)], \quad r \in (0, 1],$$

it follows that

$$-(v+\beta)'' - \frac{N-1}{r}(v+\beta)' + (\lambda m+1)(v+\beta) \geq \lambda([h(v)+mv] - [h(-\beta)+m(-\beta)]) \geq 0, \quad r \in (0, 1].$$

Denote

$$w := v + \beta.$$

Then

$$w'' + \frac{N-1}{r}w' - (\lambda m+1)w \leq 0, \quad r \in (0, 1],$$

$$w'(0) = w'(1) = 0.$$

It follows from [6, Theorem 3.5] or [12, Theorem 3 in Chapter 1] that,  $w$  cannot achieve a non-positive minimum in the interval  $(0, 1)$  unless it is constant. From (4.11), it follows that

$$\inf_{[x_0, r_0]} w(r) = \min\{w(x_0), w(r_0)\} = w(x_0) = 0.$$

This together with  $w'(x_0) = 0$  imply that

$$w(r) \equiv 0, \quad r \in [x_0, r_0].$$

However, this contradicts the fact  $w'(r) > 0$ ,  $r \in (x_0, r_0)$ . Therefore,

$$v(r) > -\beta, \quad r \in [0, 1].$$

□

In view of Lemma 4.2, (4.5) is equivalent to (4.7). So, we only need to show that

$$C_{k,\nu} \cap (\{1\} \times X) \neq \emptyset. \quad (4.13)$$

In the following, we only deal with the case ‘ $\nu = +$ ’ since the other case can be treated by the similar way.

Let  $k \geq 2$  be fixed, and let  $(\eta_n, y_n) \in C_{k,+}$  satisfy

$$\eta_n + \|y_n\|_X \rightarrow \infty.$$

It is easy to check that

$$\eta_n > 0, \quad n \in \mathbb{N}. \quad (4.14)$$

From (A3), it follows that that  $\frac{\lambda_k^r}{h'(0)} < 1$ , i.e.

$$\frac{\lambda_k^r}{f'(\beta)} < 1. \quad (4.15)$$

We shall show that

$$C_{k,+} \cap (\{1\} \times X) \neq \emptyset. \quad (4.16)$$

Assume on the contrary that  $C_{k,+} \cap (\{1\} \times X) = \emptyset$ . Then

$$C_{k,+} \subset (0, 1) \times X,$$

and accordingly,

$$0 < \eta_n < 1.$$

Thus

$$\|y_n\|_X \rightarrow \infty, \quad n \rightarrow \infty, \quad (4.17)$$

which together with Lemma 2.6 imply that

$$\|y_n\|_\infty \rightarrow \infty, \quad n \rightarrow \infty. \quad (4.18)$$

This means that  $C_{k,+}$  is unbounded in  $C[0, 1]$ !

We may assume that  $\eta_n \rightarrow \bar{\eta} \in [0, 1]$  as  $n \rightarrow \infty$ . Let

$$z_n := \frac{y_n}{\|y_n\|_\infty}.$$

Then  $\|z_n\|_\infty = 1$  and

$$\begin{cases} -z_n'' - \frac{N-1}{r}z_n' + z_n = \eta_n \frac{h(y_n)}{y_n} z_n, & r \in (0, 1), \\ z_n'(0) = z_n'(1) = 0. \end{cases} \quad (4.19)$$

From (A1)-(A3) and the definition of  $h$ , it follows that  $\frac{h(y_n(r))}{y_n(r)}$  is continuous in  $[0, 1]$  and is bounded uniformly in  $n$ . After taking subsequence if necessary, we may assume that

$$(\eta_n, z_n) \rightarrow (\bar{\eta}, z^*), \quad \text{in } \mathbb{R} \times X. \quad (4.20)$$

Here  $\|z^*\|_\infty = 1$ .

As a direct consequence of the Banach contraction mapping principle in a small neighborhood of  $\tau$ , the initial value problem

$$\begin{cases} -z'' - \frac{N-1}{r}z' + z = \bar{\eta}H(r)z, \\ z(\tau) = z'(\tau) = 0 \end{cases}$$

has a unique solution  $z \equiv 0$ . Notice that taking subsequence if necessary, we may assume that  $\frac{h(y_n)}{y_n} \xrightarrow{w} H$  in  $L^2[0, 1]$ . So, all of zeroes of  $z^*$  are simple, and accordingly  $(\bar{\eta}, z^*) \in C_{j,+}$  for some  $j \in \mathbb{N}$ .

Let

$$\tau(1, n) < \cdots < \tau(k-1, n)$$

denote the zeros of  $y_n$ , and let

$$\tau(0, n) := 0, \quad \tau(k, n) := 1.$$

Then, after taking a subsequence if necessary,

$$\lim_{n \rightarrow \infty} \tau(l, n) := \tau(l, \infty), \quad l \in \{0, 1, \dots, k-1, k\}. \quad (4.21)$$

Denote

$$J_l := (\tau(l, \infty), \tau(l+1, \infty)), \quad l \in \{0, 1, \dots, k-1\}.$$

**Claim** We claim that

$$J_l = \emptyset \quad \text{if } l \in \{0, 1, \dots, k-1\} \text{ and } l \text{ is odd,} \quad (4.22)$$

and

$$\lim_{n \rightarrow \infty} y_n(r) = +\infty \text{ uniformly in } [\tau(l, \infty) + \epsilon, \tau(l+1, \infty) - \epsilon] \text{ if } l \in \{0, 1, \dots, k-1\} \text{ and } l \text{ is even,} \quad (4.23)$$

where  $\epsilon > 0$  is small constant.

In fact, suppose on the contrary that

$$J_{l_0} \neq \emptyset \quad \text{for some } l_0 \in \{0, 1, \dots, k-1\} \text{ and } l_0 \text{ is odd.}$$

Then we have from Lemma 4.2 that

$$-\beta < y_n(r) < 0, \quad r \in (\tau(l_0, n), \tau(l_0 + 1, n)).$$

Thus, for any  $r \in (\tau(l_0, n), \tau(l_0 + 1, n))$ , it follows from (4.18) and

$$-z_n'' - \frac{N-1}{r} z_n' + z_n = \eta_n \frac{h(y_n)}{\|y_n\|_\infty}, \quad r \in (\tau(l_0, n), \tau(l_0 + 1, n)),$$

that

$$\begin{cases} -z^{*''} - \frac{N-1}{r} z^{*'} + z^* = 0, & r \in J_{l_0}, \\ z^*(\tau) = z^{*'}(\tau) = 0, \end{cases}$$

for some  $\tau \in J_{l_0}$ . This implies that

$$z^*(r) = 0 = z^{*'}(r), \quad r \in J_{l_0}.$$

However, this contradicts the fact the solution  $(\bar{\eta}, z^*) \in C_{j,+} \subset [0, 1] \times S_{j,+}$  for some  $j \in \mathbb{N}$ .

Therefore, (4.22) is true!

Obviously, (4.23) is an immediate consequence of the fact that all of the zeros of  $z^* \in S_{j,+}$  are simple and  $l \in \{0, 1, \dots, k-1\}$  is even.

Therefore, the **Claim** is true!

In the following, we shall use some idea from the proof of [7, Lemma 3.2] and the proof of main results of [3, 9] to show (4.16) is valid.

Let  $(y_n)^-$  be the negative part of  $y_n$ . Then it follows from Lemma 4.2 that  $0 \leq (y_n)^- < \beta$  since  $\eta_n \in (0, 1)$ , and consequently,

$$(z_n)^- \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Combining this with the **Claim** and using the definition of  $h$ , it concludes that

$$\begin{cases} -z^{*''} - \frac{N-1}{r} z^{*'} + z^* = \bar{\eta} f_{+\infty}(z^*)^+, & \text{a.e. } r \in (0, 1), \\ z^{*'}(0) = z^{*'}(1) = 0, \end{cases} \quad (4.24)$$

where  $(z^*)^+$  is the positive part of  $z^*$ . Now, it follows from [6, Theorem 3.5] or [12, Theorem 3 in Chapter 1] that,  $z^*$  cannot achieve a non-positive minimum in the interval  $(0, 1)$  unless it is constant. Since  $z_n(0) > 0$ , we get

$$z^*(0) \geq 0.$$

If  $z^*(0) = 0$ , then it follows from (4.24) that

$$z^* \equiv 0, \quad r \in (0, 1).$$

However, this contradicts  $\|z^*\|_\infty = 1$ . So

$$z^*(0) > 0.$$

Again, from [6, Theorem 3.5] or [12, Theorem 3 in Chapter 1] that  $z^*(r) > 0$  in  $[0, 1]$ . This means that  $z^* \in S_{1,+}$ , and therefore, since  $S_{1,+}$  is open and  $\|z_n - z^*\|_X \rightarrow 0$  as  $n \rightarrow \infty$ ,  $z_n \in S_{1,+}$  for  $n$  large enough. However, this contradicts  $z_n \in S_{k,+}$  for all  $n \in \mathbb{N}$  and  $k \geq 2$ .

Therefore, (4.16) is valid, and we may take  $v_{k,+} \in (C_{k,+} \cap (\{1\} \times X))$ . Similarly, we may take  $v_{k,-} \in (C_{k,-} \cap (\{1\} \times X))$ .

To show that  $v_{2,+}$  is decreasing in  $[0, 1]$ . Let us denote  $t_1$  ( $0 < t_1 < 1$ ) be the zero of  $v_{2,+}$ . Notice that  $v_{2,+}$  satisfies (4.3), i.e.,

$$\begin{cases} -(r^{N-1}(v_{2,+})')' = r^{N-1}[f(v_{2,+} + \beta) - (v_{2,+} + \beta)], & r \in (0, 1), \\ v_{2,+} > -\beta, \\ v_{2,+}'(0) = v_{2,+}'(1) = 0. \end{cases} \quad (4.25)$$

Combining this with (A4) and using Lemma 4.2, it concludes that

$$(r^{N-1}(v_{2,+})')' < 0, \quad t \in (0, t_1); \quad (r^{N-1}(v_{2,+})')' > 0, \quad t \in (t_1, 1). \quad (4.26)$$

This together with the boundary condition  $v_{2,+}'(0) = v_{2,+}'(1) = 0$  imply that

$$(v_{2,+})' < 0, \quad t \in (0, t_1); \quad (v_{2,+})' < 0, \quad t \in (t_1, 1).$$

Therefore,  $v_{2,+}$  is decreasing in  $[0, 1]$ .

Using the same method, with the obvious changes, we may deduce that  $v_{2,-}$  is increasing in  $[0, 1]$ .  $\square$

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